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A NOTE ON DYNAMIC STABILITY

BY
KENNETH J. ARROW
MAURICE McMANUS

TECHNICAL REPORT NO. 43

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FOR
OFFICE OF NAVAL RESEARCH

DEPARTMENT OF ECONOMICS
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1. Introduction

In a recent issue of *Econometrica*, A.C. Enthoven and K.J. Arrow [1] are interested in the following mathematical problem. Let $\varphi(A)$ be the largest of the real parts of the characteristic roots of the real, square matrix A. Then A is said to be stable if and only if $\varphi(A) < 0$. Now if A is stable, in what circumstances is DA stable, where D is diagonal? Their theorem is that if A has non-negative off-diagonal elements, this being a generalized version of the type examined by L.A. Metzler [2], then DA is stable if and only if the diagonal elements of D are all positive. The purpose of this note is to examine the same problem for certain other classes of matrices. The importance of the results to economic dynamics is discussed at the end of the paper.

2. Three Theorems on Stability.

Some preliminary remarks and lemmas will facilitate the later discussion.

Remarks: A real, square matrix M is called negative (resp. positive) quasi-definite if and only if $h'Mh$ [1] is negative (resp. positive) for every real, non-null column vector h [2]; definiteness is the special case of quasi-definiteness where M is symmetric; if M is negative (resp. positive) quasi-

[1] A prime after a vector or matrix denotes its transpose.

[2] See P.A. Samuelson [6], p. 140.

definite, so is its inverse, for $k' M^{-1}k = k'M^{-1'}M'M^{-1}k = h'M'h = h'Mh$, where $h = M^{-1}k$; the inverse of a non-singular symmetric matrix S is also symmetric, since $S^{-1} = S^{-1}I' = S^{-1}(S^{-1}S)' = S^{-1}S\,S^{-1'} = S^{-1'}$.

Lemma 1: Any real symmetric matrix S can be transformed by a real orthogonal matrix P into a diagonal matrix $D = P'SP$. The diagonal elements of D are the characteristic roots of S ; they are real, and they are all positive (resp. negative) if and only if S is positive (resp. negative) definite.

Lemma 2: A negative quasi-definite matrix is stable. The proof of the first lemma is well-known. The second follows directly from the remarks and Equation (2) below with $S = I$.

We are now in a position to state and prove

Theorem 1: If S is symmet[illegible] and A is negative quasi-definite and so stable, the SA is stable i[illegible] only if S is positive definite. The proof turns upon

Lemma 3: If M is negative quasi-definite and S is non-singular and symmetric then no characteristic root of SM has a zero real part.

Proof of Lemma 3: For any characteristic root λ , there exists a non-null vector x such that $SMx = \lambda x$. Pre-multiply both sides by $\bar{x}'S^{-1}$, where $\bar{x}$ denotes the conjugate of x :

$$(1) \qquad \bar{x}'S^{-1}SMx = \bar{x}'Mx = \lambda\,\bar{x}'S^{-1}x \quad .$$

Write $x = y + iz$, $\lambda = \alpha + i\beta$, where y, z, α and β are all real.

Equating real parts of (1):

$$(2) \qquad y' My + z' Mz = \alpha(y' S^{-1} y + z' S^{-1} z) \quad ,$$

for, by the remarks, the real coefficient of θ is zero. By hypothesis the left hand side of (2) is negative and so $\alpha \neq 0$.

Proof of the Theorem: If S is singular it is not definite and SA is singular; hence $|SA - \lambda I| = 0$ for $\lambda = 0$. Therefore, the theorem is satisfied if $|S| = 0$. Now assume that S is not singular. If it is positive definite it follows from (2) that $\alpha < 0$. This proves the sufficiency part of the theorem, though both parts are simultaneously deduced in what follows. Define:

$$A(t) = (1 - t)A - tI \quad , \qquad 0 \leq t \leq 1 \quad .$$

$A(t)$ is negative quasi-definite for all t since $h' A(t)h = (1 - t)h' Ah - th'h < 0$ because A is negative quasi-definite. Hence, by Lemma 3 , $\varphi[SA(t)] \neq 0$ for all t . By continuity, therefore, $\varphi[SA(t)]$ is either positive for all t or else negative for all t . In particular,

$$\varphi[SA(0)] < 0 \quad \text{if and only if} \quad \varphi[SA(1)] < 0 \quad .$$

But $A(0) = A$ and $A(1) = -I$. Hence, the characteristic roots of $SA(1)$ are those of $-S$. By Lemma 1 they are all real and are all negative if and only if S is positive definite. Thus $\varphi(SA)$, i.e., $\varphi[SA(0)]$, is negative if and only if S is positive definite.

The sufficiency part of Theorem 1 was stated by Samuelson [6] p. 275 [3].

Corollary: If A is negative (quasi-) definite and D is diagonal, DA is stable if and only if the diagonal elements of D are all positive.

Proof: Let D be the S matrix of Theorem 1. The diagonal elements must be positive for it to be positive definite. The corollary is the counterpart of the Enthoven-Arrow Theorem, proved for the negative (quasi-) definite type of stable matrix instead of a Metzler type.

A useful extension of the Enthoven-Arrow Theorem and the corollary of Theorem 1 above is provided by

Theorem 2: Let D be diagonal and let A have the following properties: it is stable, and DA is stable if and only if the d_{ii}, the diagonal elements of D, are all positive. Let $C = EAE^{-1}$ where E is any non-singular diagonal matrix. Then C is stable and DC is stable if and only if $d_{ii} > 0$ for all i.

The proof makes use of

Lemma 4: Any matrix B is stable if and only if $F = GBG^{-1}$ is stable for all non-singular G.

Proof: B and GBG^{-1} are similar matrices, having the same characteristic roots.

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[3] Actually he claims that, in terms of our notation, if A is negative quasi-definite then it is sufficient for the stability of, say, HA that H^{-1} (and so, by the remarks, H) be only positive quasi-definite. This, however, goes a little too far as the following counter-example shows. Let $H = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$. Then $HA = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}$ is unstable, although H is positive quasi-definite and A negative quasi-definite.

Proof of Theorem 2: If A is stable, then $C = EAE^{-1}$ is stable, by Lemma 4. Now $DC = DEAE^{-1} = E(DA)E^{-1}$ since diagonal matrices are commutative in multiplication. Hence, by Lemma 4, DC is stable if and only if DA is, that is, if and only if $d_{ii} > 0$ for all i.

Corollary: *Let* C *be such that, for some diagonal matrix* E, $A = E^{-1}CE$ *is either negative (quasi-) definite or has all off-diagonal elements non-negative and is stable. Then* DC *is stable if and only if every* d_{ii} *is positive*.

Proof: By the corollary of Theorem 1 and the Enthoven-Arrow Theorem (as the case may be) the matrix A of the corollary has the properties stipulated for A in Theorem 2.

Theorem 2 is important because if A is Metzlerian, or symmetric, or quasi-definite, $EAE^{-1} = C$ need be none of these.

A more general result covering this sort of problem is

Theorem 3: *Given* C *and a symmetric matrix* S, *let* P *be an orthogonal matrix (whose existence is guaranteed by Lemma 1) such that* $P'SP = D$, *where* D *is diagonal, and define* $\bar{C} = P'CP$. *For some diagonal* E, *let* $A = E^{-1}\bar{C}E = E^{-1}P'CPE$ *satisfy the conditions specified for* A *in Theorem 2; for instance,* A *may be Metzlerian and stable or negative quasi-definite. Then* SC *is stable if and only if* S *is positive definite*.

Proof: By Theorem 2, $D\bar{C}$ is stable if and only if all the d_{ii} are positive. But, since P is orthogonal, $D\bar{C} = P'SPP'CP = P^{-1}SCP$. Thus the roots of $D\bar{C}$ and SC are the same by dint of Lemma 4. Moreover, by Lemma 1, D is positive definite if and only if S is. It follows that SC is stable if and only if S is positive definite.

The previous results are all special cases of Theorem 3 . The Enthoven-Arrow Theorem has $S = D$, $P = E = I$; putting $S = D$ yields Theorem 2 ; finally, since $P'CP$ is negative quasi-definite if and only if C is[4/] , Theorem 1 follows if $E = I$. On the other hand, the other theorems do not cover those cases where $S \neq D$ and either A is Metzlerian or, if A is negative quasi-definite, $E \neq I$.

3. Economic Application.

The Enthoven-Arrow dynamic general equilibrium system can be written as

$$\dot{p} = Kx \; ; \quad x = Q(p-p^o) + B(p^f-p^o) \; ; \quad p^f = p + \eta \dot{p} \quad . \tag{3}$$

Here p is the column vector of all the other prices in terms of the numéraire, the dot represents differentiation with respect to time, the superscript "o" denotes "equilibrium" levels and p^f is the vector of expected future relative prices, whilst x is the vector of the corresponding excess-demands. K is the matrix of speeds of adjustment of prices to excess-demands, $Q = (dx/dp')^o$, $B = (dx/dp^{f'})^o$ and η is the matrix of extrapolative price-expectational coefficients. In the original model, K, B and η are all diagonal matrices. Eliminating x and p^f , (3) yields the set of linear differential equations,

$$\dot{p} = (I - KB\eta)^{-1} K(Q + B)(p - p^o) \quad . \tag{4}$$

4/ Cf. Samuelson [6] pp. 140-1 .

Enthoven and Arrow [1] have D and A refer to $(I - KB\eta)^{-1}$ and $K(Q + B)$, respectively. It will now be more convenient, however, to rewrite (4) as

$$(5) \qquad \dot{p} = (K^{-1} - B\eta)^{-1}(Q + B)(p - p^{o})$$

and let D (or S) refer to $(K^{-1} - B\eta)^{-1}$ and A (or C) to $(Q + B)$. The Samuelsonian expectationless model is a special case with $\eta = 0$. Denote the elements of matrices by the corresponding small letters with appropriate row and column suffixes. Then the corollary to Theorem 2 tells us that if $E^{-1}(Q + B)E$ is either negative (quasi-) definite or Metzlerian and stable for some E (possibly, of course, $E = I$), the expectationless system is stable if and only if all the k_{ii} are positive - the usual behavior. Moreover, for the same $(Q + B)$, the introduction of the expectational coefficients does not upset stability if and only if $1/k_{ii} > b_{ii}\eta_{ii}$ for all i.

Although the same conclusions do not hold for arbitrary stable $(Q + B)$ 5/, it is fortunate that some of the types that are covered by the theorems are of particular interest to the economist. Q is symmetric if income effects are either absent or else symmetrical. Alternatively, if the matrix has non-negative off-diagonal elements, it means that the system contains no "gross complements" 6/. When all is said and done, however, we must expect to find

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5/ For example, $\begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$ is stable, yet $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$ is unstable and $\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$ is stable.

6/ The adjective "gross" denotes that income effects are taken into account. The phrase is due to J. Mosak [4].

in any actual economy substantial, asymmetrical income effects and a goodly sprinkling of gross complementary. It is desirable, therefore, to try to widen the classes of matrices about which useful statements about stability can be made. An important stride in this direction was made by Samuelson [6] in his consideration of non-symmetric, negative quasi-definite matrices, and the E transformation provides a way of generating further types out of the two or three basic ones. If, for example, A is Metzlerian and the elements of E are of mixed signs, then EAE^{-1} is of that general class which has been examined in detail by M. Morishima [3].

An additional interest in Theorems 1 and 3 lies in the fact that S need not be diagonal, only symmetric. This has an application in terms of the above model. The speed of price adjustment in any one market may depend upon the excess-demands in other markets as well as in its own, as has been argued cogently by D. Patinkin [5] p. 157 ; it is quite feasible that some excess-demands partly depend upon expected prices in other markets; it may even be that expected prices are influenced by what is happening to several different current prices. Any combination of these makes $(I - KB\eta)^{-1}$ or $(K^{-1} - B\eta)^{-1}$ non-diagonal in general.

Samuelson [6] pp. 274-5 subjects the original variables, p and x, to a contragredient transformation $\bar{p} = c'p$; $x = c\bar{x}$. The system (5) becomes

$$\dot{\bar{p}} = c'(K^{-1} - B\eta)^{-1} c\, c^{-1}(Q + B)c'^{-1} (\bar{p} - \bar{p}^{o}) \quad . \tag{6}$$

The transformation of Lemma 1 as used in Theorem 3 is a special case where c is the orthogonal P. Using this interpretation, SC and $D\bar{C}$ would refer to the <u>same</u> economic system, but one in which two different definitions of what constitute the given number of commodities are considered.

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